DE-GROOT-LIKE DUAL OF PRETOPOLOGICAL SYSTEMS

Chernikava Alena
Doctoral Degree Programme (1), FEEC BUT
E-mail: xcerni07@stud.feec.vutbr.cz

Supervised by: Martin Kovár
E-mail: kovar@feec.vutbr.cz

Abstract: In this paper we study a slight modification of the De Groot's dualization construction for the topological structures with the preframe structure (in the sense of Banaschewski) of opens. We also present some counterexamples, contributing to the discussion regarding the possibility of obtaining similar results as there are already known for the classical topological spaces.

Keywords: Preframe, Frame, Pretopological system, de Groot-like dual, Topology

1 INTRODUCTION

Modern topological methods are widely used in many recent scientific applications, including theoretical computer science, formal concept analysis, digital image analysis and processing, causal quantum structures and study of qualitative properties of certain differential equations. By means of these highly theoretical disciplines, topological results are also applied in the theory of parallel computation and concurrent processes, quantum algorithms, analysis of digital images in tomography, microscopy or echolocation, electron holography, quantum gravity and the theory of quantum topological insulators.

One of the most important aspects of the studied topological properties having some relationship to the above mentioned applications there is the (well-known) construction of the de Groot dual. Recall that for a given topological space (X,τ), a topology τ^d, generated by the family of all compact saturated sets used as its closed base, is called the de Groot dual of the original topology τ. Its importance for applications (especially in theoretical computer science) is witnessed by the paper of Jimmie Lawson and Michael Mislove included in the monograph [12]. J. Lawson and M. Mislove stated there a problem, whether the sequence containing the iterated duals of τ is infinite or the process of taking duals terminates after finitely many steps with two topologies which are dual to each other. The problem was solved by Martin Kovár in 2001. He proved that for any topology it holds $\tau^{dd} = \tau^{dddd}$ [9]. In 2004 the result was improved by the same author to its (so far) final form $\tau^d = (\tau \vee \tau^{dd})^d$ [11]. Note that from this result it also follows that $\tau^d \subseteq \tau^{ddd}$ for any topology τ. It should be also noted that in [9] M. Kovár stated several natural questions regarding the dual topologies. Some of them were studied by Tomoo Yokoyama in his recent paper [14].

The questions of J.Lawson and M.Mislove related to the de Groot dual arise from study of various semantic models in the theoretical computer science, where the dual and the patch topologies are an important tools of investigation. Another interesting direction of research was introduced by Bernhard Banaschewski [1], who replaced the usual frame structure by a more general, partially ordered structure called *preframe*, where the suprema exist for all non-empty up-directed subcollections. Taking some inspiration from the "classic" results of J. Lawson, M. Mislove, M. Kovár, T. Yokoyama, and from B. Banaschewski's preframe structure of opens of pretopological systems, we investigate the possibility of a construction analogous to the de Groot dual, but in a new, modified setting. A possible range of applications could lie in improvements of the efficiency of some topological algorithms and investigation of the properties of certain causal structures, applicable in quantum gravity and the theory of quantum topological insulators.

The results of this article are based on two recent joint papers and conference contributions of the author with M. Kovár. Because of a limited range, the proofs are not included. However, in case of interest, the reader is referred to [3] and [4].

2 DE GROOT DUAL IN COMPACTLY LOCALIC LOGIC

We will start with recalling some key notions and making several useful denotations. By $\mathbf{2} = \{\bot, \top\}$ we mean the *Serpiński frame*, consisting of the two elements $\top$ – the top and $\bot$ – the bottom. Let (X,τ) be a topological space. By τ^d we denote the topology generated by the compact saturated sets in (X,τ) used as its closed base. This topology is also often called *co-compact* and the operator d is now called the *de Groot dual*. For a partially ordered set, or shortly a *poset* $(P,\leq)$, the *weak* topology is defined by taking the principal lower sets $\downarrow\{x\}$, for $x \in P$, as the closed subbase. Similarly, the *weak*^{d} topology is defined by taking the principal upper sets $\uparrow\{x\}$, for $x \in P$, as the closed subbase for a topology on P. It is well-known that in a locale $(X,A,\vdash)$, the set of points X may be represented as a family of all frame morphisms $: A \to \mathbf{2}$ and the relation $\vdash$ is defined by $x \vdash a \Leftrightarrow x(a) = \top$ for $x \in X$ and $a \in A$. The Hofmann-Mislove theorem says that there is 1-1 correspondence between the compact saturated sets in $(X,A,\vdash)$ and the functions from A to $\mathbf{2}$ that preserve directed joins and finite meets. Taking these functions as points and the elements of A as opens, we obtain a new structure $(X',A,\vdash')$ that redistributes the logic: The localic points are replaced by the compacts sets and the new relation $\vdash'$ preserves directed joins as well as finite joins on both sides. On the other hand, it should be noted that $(X',A,\vdash')$ need not be a topological system in the usual sense but a structure slightly different. We will specify it in a more detail.

Recall that a poset A is a *preframe*, if A is closed under (non-empty) directed joins and finite meets (including the meet of the empty set), such that the binary meets distribute over the directed joins. We write $\top = \bigwedge\emptyset$ and $\bot = \bigvee\emptyset$. A simple example of a preframe which is not a frame is given by the poset $P = \{\bot, 0\} \cup \mathbb{N}$ on the figure:

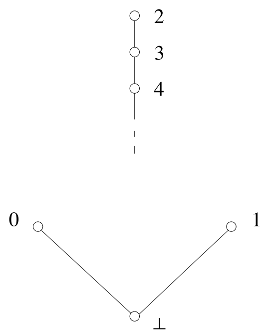

Figure 1.

Let A be a preframe, X a set, $\vdash \subseteq X \times A$. We write $x \vdash a$ for $(x,a) \in \vdash$ and say "x satisfies a". Let the following conditions are satisfied:

(i) If $B \subseteq A$ is non-empty and directed, then $(x \vdash \bigvee B) \Leftrightarrow (x \vdash b$ for some $b \in B)$.

(ii) If $C \subseteq A$ is non-empty and finite, then $(x \vdash \bigwedge C) \Leftrightarrow (x \vdash c$ for every $c \in C)$.

Then we say that the triple $(X,A,\vdash)$ is a *pretopological system*. The elements of A we call, similarly as in topological systems, *opens*. If $A \subseteq 2^X$ is ordered by the inclusion, $\emptyset, X \in A$ and the relation $\vdash$ is $\in$, then A is called a *pretopology* and $(X,A,\vdash)$ is a *pretopological space*.

Let A be a poset. We denote by $\langle A \to \mathbf{2}\rangle \subseteq \mathbf{2}^A$ the set of all functions $A \to \mathbf{2}$ that preserve the non-empty directed joins and finite meets, whenever they exist. The elements of $\langle A \to \mathbf{2}\rangle$ we will call *morphisms*.

Proposition 2.1 *The poset $\langle A \to \mathbf{2}\rangle$ forms a preframe of all morphisms of A to $\mathbf{2}$.*

Example 2.2 *Let A be a preframe. We put $x \vdash a$ if and only if $x(a) = \top$ for $x \in \langle A \to \mathbf{2}\rangle$ and $a \in A$. Then $(\langle A \to \mathbf{2}\rangle, A, \vdash)$ is a pretopological system.*

It can be shown that similarly as in locales, the pretopological system constructed in the previous example is fully determined by A. Thus we say that a pretopological system $(X,A,\vdash)$ is a *compactly localic* if $X = \langle A \to \mathbf{2}\rangle$ and $x \vdash a$ if and only if $x(a) = \top$ for $x \in \langle A \to \mathbf{2}\rangle$ and $a \in A$. Let us denote, similarly as in [10], $\mathrm{int}_X(a) = \{x | x \in X, x \models a\}$ for every $a \in A$. We say that $K \subseteq X$ is *compact* in a pretopological system $(X,A,\vdash)$ if for every directed $B \subseteq A$ with $K \subseteq \mathrm{int}_X(\bigvee B) = \bigcup_{b \in B} \mathrm{int}_X(b)$, there is some $a \in B$ such that $K \subseteq \mathrm{int}_X(a)$. We say that $S \subseteq X$ is *saturated*, if S is an intersection of the sets $\mathrm{int}_X(b)$, $b \in B$ for some $B \subseteq A$.

One can easily check that the notions of compactness and saturation in pretopological systems slightly differ from their counterparts in topological systems if A is not a frame (although the sets $\mathrm{int}_X(a)$, $a \in A$ generate some underlying topology on X). If A is not a frame, the previously introduced notions need not coincide with compactness and saturation related to this topology. Although it is obviously possible to define a dualization of a general pretopological system, it is not so simple to choose the best construction from several possibilities since they are not easily comparable with the classical topological case.

Now, let $(X,A,\vdash)$ be a compactly localic pretopological system. For any $x \in X$ and any $y \in \langle A \to \mathbf{2}\rangle$ we say that x is *independent* on y and write $x \models y$ if there is some $a \in A$ such that $y(a) = \top$ and $x \nvdash a$. Under the inspiration of the construction of the de Groot dual of a topological space (see e.g. [12], [2], [5] or [8]) and constructions studied in [10], by the dualization of a compactly localic pretopological system $(X,A,\vdash)$ we mean the triple $(X, \langle A \to \mathbf{2}\rangle, \models)$. We will see that under the condition that A is a frame, $(X, \langle A \to \mathbf{2}\rangle, \models)$ will correspond to its topological counterpart.

Theorem 2.3 *Let A be a frame and $(X,A,\vdash)$ be a compactly localic pretopological system. Then $(X, \langle A \to \mathbf{2}\rangle, \models)$ is also a pretopological system.*

Lemma 2.4 *Let $(X,A,\vdash)$ be a compactly localic pretopological system. Then the non-empty compact saturated sets in $(X,A,\vdash)$ are the sets of the form $\uparrow\{x\} = \{y | y \in \langle A \to \mathbf{2}\rangle, y \geq x\}$, where $x \in X$.*

Theorem 2.5 *Let A be a frame, $(X,A,\vdash)$ be a compactly localic pretopological system. Then the topology on X induced by $\langle A \to \mathbf{2}\rangle$ is dual to the topology on X induced by A in the usual sense and it equals to its weak^d topology.*

On the other hand, if A is a more general preframe than a frame, the triple $(X, \langle A \to \mathbf{2}\rangle, \models)$ representing the dual of $(X,A,\vdash)$ even need not be a pretopological system and if so, it need not be compactly localic. Hence, the sequences of iterated dualizations are not possible in general in this setting. One possible idea how to fix this problem could be modifying the underlying set of points of the dual pretopological system. This idea leads to the following natural definition.

By a *compactly localic dualization* of a compactly localic pretopological system $(X,A,\vdash)$ we mean the pretopological system $(X', \langle A \to \mathbf{2}\rangle, \Vdash)$, where $X' = \langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ and $(u \Vdash y) \Leftrightarrow (u(y) = \top)$

for $u \in X'$ and $y \in \langle A \to \mathbf{2} \rangle$. Now the iterated compactly localic dualizations exist for any compactly localic pretopological system, and they are fully represented by sequence of the posets of their opens: $\langle A \to \mathbf{2} \rangle$, $\langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle$, $\langle \langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle \to \rangle, \ldots$, etc.

3 DUALIZATIONS FOR THE POSETS OF OPENS

Now we will concentrate on the preframe structure of the opens of the pretopological counterpart of the de Groot dual. As we have shown in the previous section, the opens of the dual may be represented as certain maps from A to the Sierpiński frame $\mathbf{2}$, where A is the poset representing the opens of the original pretopological system.

Let A be a poset. We denote by $h_A : A \to \langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle$ a mapping for which $h_A(a)(x) = x(a)$ for every $x \in \langle A \to \mathbf{2} \rangle$. The following theorem holds:

Theorem 3.1 *Let A be a poset. Then $h_A : A \to \langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle$ is a morphism.*

On the other hand, we have the following two counterexamples; the corresponding posets are given by their Hasse diagrams on the figures:

Example 3.2 *There exist a preframe A such that h_A is not an epimorphism.*

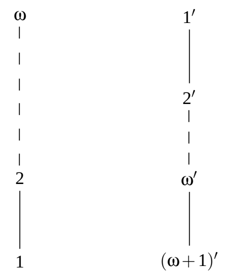

Figure 2.

Example 3.3 *There exist a preframe A such that h_A is not a monomorphism.*

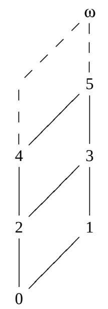

Figure 3.

The positive results can be reached especially for the finite case:

Theorem 3.4 *Let A be a finite preframe. Then $h_A : A \to \langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle$ is an isomorphism.*

Corollary 3.5 *Let A be a finite poset. Then its iterated duals, $\langle A \to \mathbf{2} \rangle$ and $\langle \langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle \to \mathbf{2} \rangle$, are isomorphic.*

4 CONCLUSION

We have successfully defined an analogue of De Groot dual for compactly localic pretopological systems and in a more general approach, also for any poset. For finite posets we also have an adequate counterpart of M. Kovár's result $\tau^d \subseteq \tau^{ddd}$ proved for the general topological spaces (as it is shown in Proposition 3.4).

However, the counterexamples Example 3.2 and Example 3.3 show that a requested result for general (potentially infinite) posets cannot be reached in this simple form. The reason obviously lies in the fact that – unlike in the topological case – the points of the corresponding pretopological systems are modified in each step of taking the dual. And this should be our starting point of our further considerations.

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